

Expected Value: Example II

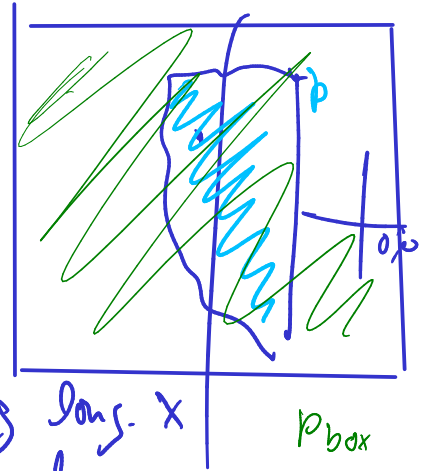
- What is the expected snowfall in Illinois?

$$E[\underline{\text{Snow}}] = \int_{\mathcal{R}} \text{snow}(x) p(x) dx$$

x : longitude

$\text{snow}(x)$: how much snow along long. x

$p(x)$: how likely is a point on long. x
to be in Illinois



$$\iint_{\mathcal{R}} 1 \cdot \begin{cases} 1 & \text{if } (x,y) \text{ is in } I \\ 0 & \text{if } (x,y) \text{ is not} \end{cases} = \text{Area}$$

$$\mathbb{E}[\text{Snow}] = \int_{\mathcal{R}} \int_{\mathcal{R}} \text{Snow}(x,y) p(x,y) dx dy$$

$\text{Snow}(x,y)$: how much snow at long x and lat. y

$p(x,y)$: how likely is it that x and lat y is in $\mathcal{L}$

$$\rightarrow \mathbb{E}[1] = \int_{\mathcal{R}} \int_{\mathcal{R}} 1 p(x,y) dx dy = 1$$

$$\tilde{p}(x,y) = \begin{cases} 1 & \text{if } (x,y) \text{ is in } \mathcal{L} \\ 0 & \text{if not} \end{cases}$$

$$p(x,y) = C \cdot \tilde{p}(x,y)$$

If we can compute approximate expected values.

- use that to compute $E_{\tilde{p}}[1]$ using

$\tilde{p}$. Use $E[1] = 1$ to find C

$$C = \frac{1}{E_{\tilde{p}}[1]}$$

- Use that $E[\text{Snow}] = \iint_{\mathbb{R}^2} \text{Snow}(x,y) p(x,y) dx$

Tool: Law of Large Numbers

Terminology:

- *Sample*: A random number x_i whose values follow a distribution $p(x)$.

In words:

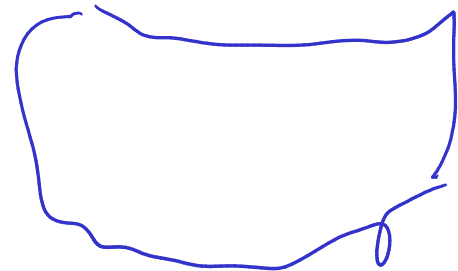
- As the number of samples $N \rightarrow \infty$, the average of samples converges to the expected value with probability 1.

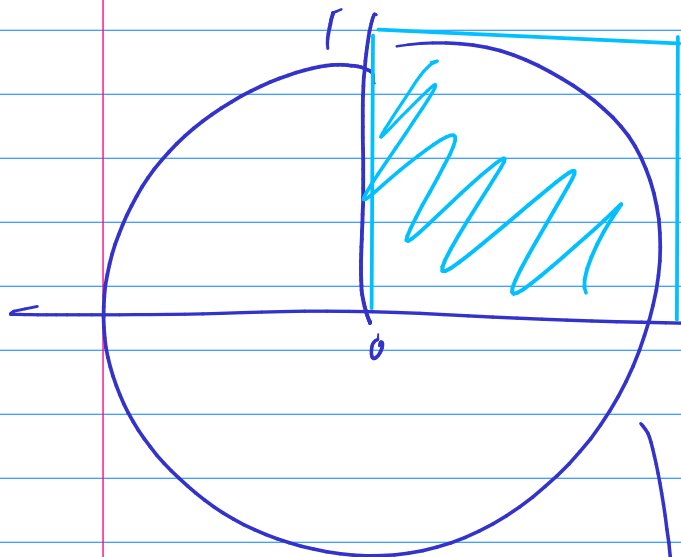
In symbols:

$$P\left[\lim_{N \rightarrow \infty} \frac{1}{N} \left(\sum_{n=1}^N x_i \right) = E[X]\right] = 1.$$

Or:

$$E[X] \approx \frac{1}{N} \left(\sum_{n=1}^N x_i \right)$$





π = Area of unit circle

$$\tilde{p}(x,y) = \begin{cases} 1 & \text{if } (x,y) \text{ in } A \\ 0 & \text{if not} \end{cases}$$

(not yet PD)

$$E_p[1] = \text{Area}(\text{box})$$

$$E_{\text{box}}[1 \cdot \tilde{p}] = \text{Area of } A$$

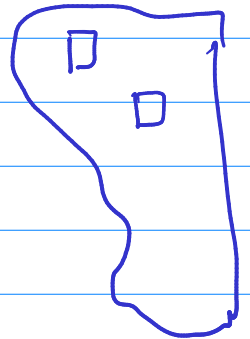
$$\text{Area} \cdot p(x,y) = \int \tilde{p}(x,y)$$

$$p(x,y) = \frac{1}{\text{Area of } A} \cdot \tilde{p}(x,y)$$

$$\bar{E}_p(S_{\text{box}}) = \iint_{\text{box}} \underbrace{S_{\text{box}}(x,y)} \underbrace{p(x,y)} dx dy \quad \leftarrow$$

$$= \underbrace{\text{Area}(\text{box})} \int_{\text{box}} \int_{\text{box}} \underbrace{S_{\text{box}}(x,y)p(x,y)} \underbrace{p_{\text{uniform, box}}(x,y)} dx dy$$

$$= \text{Area}(\text{box}) \cdot \bar{E}_{p_{\text{unif, box}}} [S_{\text{box}} \cdot p]$$



Sampling: Approximating Expected Values

Integrals and sums in expected values are often challenging to evaluate.

- How can we approximate an expected value?

Idea: Draw random samples. Make sure they are distributed according to $p(x)$.

- What is a **Monte Carlo** method?

Sampling II: Approximating Expected Values

- What if I *can't* sample from $p(x)$?

Idea: Draw uniformly distributed random samples.

Demo: Computing π using Sampling

Demo: Errors in Sampling

Sampling: Error

The **Central Limit Theorem** states that with

$$S_n := x_1 + x_2 + \cdots + x_n$$

for the (x_i) independent and identically distributed we have that

$$\frac{S_n - n E[x_i]}{\sqrt{\sigma^2[x_i]n}} \rightarrow \mathcal{N}(0, 1),$$

i.e. that term approaches the normal distribution. Or, short and imprecise:

$$\left| \frac{1}{n} S_n - E[x_i] \right| = O\left(\frac{1}{\sqrt{n}} \right).$$